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LETTER TO THE EDITOR

On the reduction and some new exact solutions of the non-linear Dirac and Dirac–Klein–Gordon equations

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Abstract. New ansätze for spinor fields are suggested. Using these, we construct multiparameter families of exact solutions of the non-linear many-dimensional Dirac and Dirac–Klein–Gordon equations, some solutions including arbitrary functions.

In this letter we have constructed new families of exact solutions of the following equations:

$$(\gamma_\mu p^\mu - \lambda(\bar\psi\psi)^\kappa)\psi(x) = 0 \qquad \mu = 0, 1, 2, 3 \tag{1}$$

$$\begin{aligned} &[\gamma_\mu p^\mu - (\lambda_1|u|^{\kappa_1} + \lambda_2(\bar\psi\psi)^{\kappa_2})]\psi(x) = 0 \\ &[p_\mu p^\mu = (\mu_1|u|^{\kappa_1} + \mu_2(\bar\psi\psi)^{\kappa_2})^2]u(x) = 0 \end{aligned} \tag{2}$$

where γ_μ are (4×4)-Dirac matrices, $\psi = \psi(x)$ is a four-component spinor, $u = u(x)$ is a complex scalar function, $p_0 = \mathrm{i}\partial/\partial x_0$, $p_a = -\mathrm{i}\partial/\partial x_a$, $a = \overline{1,3}$; λ, κ, λ_i, μ_i and κ_i are constants. Hereafter we use the summation convention.

Solutions obtained by us differ from those already known in the literature [1-8]. These solutions can be useful in the relativistic quantum field theory.

To construct exact solutions of equation (1) we use the following ansätze:

$$\begin{aligned} &\psi(x) = [\mathrm{i}g_1(\omega) + \gamma_4 g_2(\omega) - (\mathrm{i}f_1(\omega) + \gamma_4 f_2(\omega))\gamma_\mu p^\mu\omega]\chi \\ &\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3 \end{aligned} \tag{3}$$

$$\begin{aligned} \psi(x) = [&G_1(\omega_1, \omega_2) + \mathrm{i}(\gamma_\mu a^\mu + \gamma_\mu d^\mu)G_2(\omega_1, \omega_2) \\ &+ \mathrm{i}(\gamma_\mu b^\mu)F_1(\omega_1, \omega_2) + (\gamma_\mu a^\mu + \gamma_\mu d^\mu)(\gamma_\nu b^\nu)F_2(\omega_1, \omega_2)]\chi \end{aligned} \tag{4}$$

where $\omega = \omega(x)$ are scalar functions satisfying conditions of the form

$$p_\mu p^\mu\omega + A(\omega) = 0 \qquad (p_\mu\omega)(p^\mu\omega) + B(\omega) = 0 \tag{5}$$

where f_i, g_i, F_i, G_i, A and B are arbitrary differentiable functions, $\omega_1 = a_\mu x^\mu + d_\mu x^\mu$, $\omega_2 = b_\nu x^\nu$ and χ is an arbitrary constant spinor. Hereafter a_μ, b_μ, c_μ and d_μ are arbitrary real parameters satisfying the following conditions:

$$-a_\mu a^\mu = b_\mu b^\mu = c_\mu c^\mu = d_\mu d^\mu = -1$$

$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Substitution of ansätze (3) and (4) into the initial equation (1) leads to the following systems of differential equations for unknown functions f_i, g_i, F_i, G_i:

$$\begin{aligned} &B\dot{f}_1 + Af_1 = \tilde{\lambda}[g_1^2 - g_2^2 + B(f_1 - f_2^2)]^\kappa g_1 \\ &\dot{g}_1 = -\tilde{\lambda}[g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^\kappa f_1 \\ &\dot{g}_2 = \tilde{\lambda}[g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^\kappa f_2 \\ &B\dot{f}_2 + Af_2 = -\tilde{\lambda}[g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^\kappa g_2 \qquad \tilde{\lambda} = \lambda(\bar{\chi}\chi)^\kappa \\ &\dot{f}_i = \mathrm{d}f_i/\mathrm{d}\omega \qquad \dot{g}_i = \mathrm{d}g_i/\mathrm{d}\omega \qquad i = \overline{1,2} \end{aligned} \tag{6}$$

$$\begin{aligned} &F'_{\omega_2} = -\tilde{\lambda}[(G^1)^2 - (F^1)^2]^\kappa G^2 \qquad G^1_{\omega_2} = -\tilde{\lambda}[(G^1)^2 - (F^1)^2]^\kappa F^1 \\ &G^1_{\omega_1} + F^2_{\omega_2} = -\tilde{\lambda}[(G^1)^2 - (F^1)^2]^\kappa G^2 - G^2_{\omega_2} + F^1_{\omega_1} = \tilde{\lambda}[(G^1)^2 - (F^1)^2]^\kappa F^2. \end{aligned} \tag{7}$$

Not going into details of the integration of systems (5) and (6) we shall write down exact solutions of the non-linear Dirac equation (1) obtained through the substitution of expressions for f_i, g_i into ansatz (3)

(i) $\kappa < \frac{1}{4}$

$$\begin{aligned} \psi(x) = \omega^{-1/2\kappa}\{&\mp(1-4\kappa)^{1/2}(-\mathrm{i}C_1 + \gamma_4 C_2) + (C_1 - \mathrm{i}\gamma_4 C_2) \\ &\times[(\gamma b)(by) + (\gamma c)(cy) + (\gamma d)(dy)]\omega^{-1}\}\chi \\ &\omega = [(by)^2 + (cy)^2 + (dy)^2]^{1/2} \qquad C_j = \text{constant} \end{aligned} \tag{8}$$

and the condition holds

$$\pm(1-4\kappa)^{1/2} - 2\kappa\tilde{\lambda}[4\kappa(C_1^2 - C_2^2)]^\kappa = 0$$

(ii) $\kappa > \frac{1}{4}$

$$\begin{aligned} \psi(x) = \omega^{-1/2\kappa}\{&\mp(4\kappa-1)^{1/2}(-\mathrm{i}C_1 + \gamma_4 C_2) + (C_1 - \mathrm{i}\gamma_4 C_2) \\ &\times[(\gamma a)(ay) - (\gamma b)(by) - (\gamma c)(cy)]\omega^{-1}\}\chi \\ &\omega = [(ay)^2 - (by)^2 - (cy)^2]^{1/2} \qquad C_j = \text{constant} \end{aligned} \tag{9}$$

and the condition holds

$$\pm(4\kappa-1)^{1/2} - 2\kappa\tilde{\lambda}[4\kappa(C_1^2 - C_2^2)]^\kappa = 0$$

(iii) $\kappa > \frac{1}{6}$

$$\begin{aligned} \psi(x) = \omega^{-1/2\kappa}[&\mp(6\kappa-1)^{1/2}(-\mathrm{i}C_1 + \gamma_4 C_2) + (C_1 - \mathrm{i}\gamma_4 C_2) \\ &\times(\gamma y)\omega^{-1}]\chi \qquad \omega(yy)^{1/2} \qquad C_j = \text{constant} \end{aligned} \tag{10}$$

and the condition holds

$$\pm(6\kappa-1)^{1/2} - 2\kappa\tilde{\lambda}[6\kappa(C_1^2 - C_2^2)]^\kappa = 0$$

(iv) $\kappa \in \mathbb{R}^1$

$$\begin{aligned} &\psi(x) = \{\mathrm{i}g_1(\omega) + \gamma_4 g_2(\omega) + (f_1(\omega) - \mathrm{i}\gamma_4 f_2(\omega))[\gamma b + (\gamma a + \gamma d)\dot{F}(ay + dy)]\}\chi \\ &f_1 = C_1\cosh[\tilde{\lambda}(C_3^2 - C_1^2)^\kappa\omega + C_2] \qquad f_2 = C_3\cosh[\tilde{\lambda}(C_3^2 - C_1^2)^\kappa\omega + C_4] \\ &g_1 = C_1\sinh[\tilde{\lambda}(C_3^2 - C_1^2)^\kappa\omega + C_2] \qquad g_2 = C_3\sinh[\tilde{\lambda}(C_3^2 - C_1^2)^\kappa\omega + C_4] \\ &\omega = by + F(ay + dy) \qquad C_j = \text{constant} \end{aligned} \tag{11}$$

where F is an arbitrary differentiable function;

$$
\begin{aligned}
&\psi(x)=[\mathrm{i}g_1(\omega)+\gamma_4 g_2(\omega)+(f_1(\omega)-\mathrm{i}\gamma_4 f_2(\omega))(\gamma a)]\chi\\
&f_1=C_1\sin[\tilde\lambda(C_1^2-C_3^2)^\kappa\omega+C_2] \qquad f_2=C_3\cos[\tilde\lambda(C_1^2-C_3^2)^\kappa\omega+C_4]\\
&g_1=C_1\cos[\tilde\lambda(C_1^2-C_3^2)^\kappa\omega+C_2] \qquad g_2=C_3\sin[\tilde\lambda(C_1^2-C_3^2)^\kappa\omega+C_4]\\
&C_j=\text{constant} \qquad \omega=ay
\end{aligned}
\tag{12}
$$

(v) $\kappa=1/m \qquad m=2,3$

$$
\begin{aligned}
\psi(x)&=(1+\theta^2\omega^2)^{-(m+1)/2}[\mathrm{i}C_1+\gamma_4C_2-\theta(C_1+\mathrm{i}\gamma_4C_2)]\\
&\quad\times\begin{cases}[(\gamma a)(ay)-(\gamma b)(by)-(\gamma c)(cy)] & m=2\\ \gamma y & m=3\end{cases}
\end{aligned}
\tag{13}
$$

$$
\omega=\begin{cases}[(ay)^2-(by)^2-(cy)^2]^{1/2} & m=2\\ (yy)^{1/2} & m=3\end{cases}
$$

and the condition holds

$$(m+1)\theta-\tilde\lambda(C_1^2-C_2^2)^{1/m}=0.$$

In the formulae (8)-(13) the following notations were used:

$$
\begin{aligned}
&ay\equiv a_\mu y^\mu \qquad \gamma a\equiv\gamma_\mu a^\mu \qquad \gamma y\equiv\gamma_\mu y^\mu \qquad \mu=0,1,2,3\\
&y_\mu=x_\mu+\theta_\mu \qquad \theta_\mu=\text{constant} \qquad \tilde\lambda=\lambda(\bar\chi\chi)^\kappa.
\end{aligned}
\tag{14}
$$

If $g_2\equiv f_2\equiv 0$, $\omega=x_\mu x^\mu$ then (3) coincides with the ansatz suggested by Heisenberg in [1]. That is why exact solutions of the equation (1) obtained with the help of the Heisenberg ansatz in [2-4] belong to classes (10) and (13).

It was Gürsey who showed that under $k=\frac{1}{3}$ equation (1) is conformally invariant [9]. This makes it possible to construct new families of exact solutions using the solution generation technique (see [6]). As is shown in [6] the formula of generating solutions by final transformations of the four-parameter special conformal group has the form

$$
\begin{aligned}
&\psi_2(x)=\sigma^{-2}(x)[1-(\gamma x)(\gamma\alpha)]\psi_1(x')\\
&x'_\mu=(x_\mu-\alpha_\mu xx)\sigma^{-1}(x)\\
&\sigma(x)=1-2\alpha x+(\alpha\alpha)(xx) \qquad \alpha_\mu=\text{constant} \qquad \mu=\overline{0,3}.
\end{aligned}
\tag{15}
$$

Using (9)-(13) under $k=\frac{1}{3}$ as $\psi_1(x)$ one can obtain multi-parameter families of solutions of the non-linear Dirac equation (1) which are invariant under the conformal group C(1, 3).

System (7) proved to be an integrable one. Substituting its general solution into the ansatz (4) we obtain a multi-parameter family of exact solutions of the equation (1) depending on four arbitrary functions

$$
\begin{aligned}
\psi(x)=\Bigg\{&\phi_1\cosh(\phi bx)+\phi_2\sinh(\phi bx)+\mathrm{i}(\gamma a+\gamma d)[(bx/2\phi)\dot\phi\\
&\times(\phi_2\cosh(\phi bx)+\phi_1\sinh(\phi bx)+\phi_3\cosh(\phi bx)+\phi_4\sinh(\phi bx))]\\
&+\mathrm{i}(\gamma b)(\phi\sinh(\phi bx)+\phi_2\cosh(\phi bx))+(\gamma a+\gamma d)(\gamma b)\\
&\times\left[\left(\frac{\phi_2\dot\phi}{\phi^2}-\frac{\phi_1}{\phi}-\frac{\phi_2\dot\phi}{2\phi}bx\right)\sinh(\phi bx)+\left(\frac{\phi_1\dot\phi}{2\phi^2}-\frac{\phi_2}{\phi}\right.\right.\\
&\left.\left.-\frac{\phi_1\dot\phi}{2\phi}bx\right)\cosh(\phi bx)+\phi_3\sinh(\phi bx)+\phi_4\cosh(\phi bx)\right]\Bigg\}\chi
\end{aligned}
\tag{16}
$$

$$\dot\phi = \frac{d\phi}{d\omega} \qquad \dot\phi_i = \frac{d\phi_i}{d\omega} \qquad i = \overline{1,2}$$

where

$$\gamma a \equiv \gamma_\mu a^\mu \qquad \gamma b \equiv \gamma_\mu b^\mu \qquad bx \equiv b_\mu x^\mu$$

$$\phi(\omega) = -\lambda(\bar\chi\chi)^\kappa[(\phi_1(\omega))^2 - (\phi_3(\omega))^2]^\kappa \qquad \phi_1, \ldots, \phi_4$$

are arbitrary differentiable functions of $\omega = a_\mu x^\mu + d_\mu x^\mu$.

We note that it is not difficult to construct an explicit form of the energy-momentum tensor

$$T_{\mu\nu} = \tfrac{1}{2}\mathrm{i}(\bar\psi\gamma_\mu\psi_\nu - \bar\psi_\nu\gamma_\mu\psi) + g_{\mu\nu}\mathscr{L}$$

$$\mathscr{L} = \tfrac{1}{2}\mathrm{i}(\bar\psi\gamma_\mu\psi_\mu - \bar\psi_\mu\gamma_\mu\psi) - \frac{\lambda}{k+1}(\bar\psi\psi)^{k+1}$$

corresponding to obtained solutions. For the solutions (8)-(10) one has

$$T_{\mu\nu} \overset{\omega\to+\infty}{\sim} \theta_{\mu\nu}\omega^{-(\kappa+1)/\kappa} \qquad \theta_{\mu\nu} = \text{constant}. \tag{17}$$

If $0 < \kappa < \frac{1}{4}$ (for (8)) or $\kappa > \frac{1}{6}$ (for (10)) then $T_{\mu\nu}$ has a non-integrable singularity in the point $x_\mu = -\theta_\mu$, in other points of the Minkowsky space $R(1,3)$ expression (17) being integrable. In the case $\kappa > \frac{1}{4}$ (for (9)) $T_{\mu\nu}$ has a singularity on the cone

$$ay = \pm[(by)^2 + (cy)^2]^{1/2}$$

while at other points it is integrable.

Ansätze (3), (4) proved to be very useful while constructing solutions of the system (2). We shall write down some of the families of exact solutions obtained, omitting intermediate calculations.

(i) $\kappa_1 > 1$, $\kappa_2 > \frac{1}{4}$

$$\begin{aligned}\psi(x) &= \omega^{-1/2\kappa_2}\{\mp(4\kappa_2-1)^{1/2}(-\mathrm{i}C_1 + \gamma_4 C_2) + (C_1 - \mathrm{i}\gamma_4 C_2)\\ &\quad\times[(\gamma a(ay) - (\gamma b)(by) - (\gamma c)(cy)]\omega^{-1}\}\chi\\ u(x) &= E\omega^{-1/\kappa_1} \qquad C_i = \text{constant}\\ \omega &= [(ay)^2 - (by)^2 - (cy)^2]^{1/2}\end{aligned} \tag{18}$$

and the following conditions hold:

$$(1-\kappa_1)\kappa_1^{-2} + \{\mu_1|E|^{\kappa_1} + \mu_2(\bar\chi\chi)^{\kappa_2}[(C_1^2 - C_2^2)4\kappa_2]^{\kappa_2}\}^2 = 0$$

$$\pm(4\kappa_2-1)^{1/2} - 2\kappa_2\{\lambda_1|E|^{\kappa_1} + \lambda_2(\bar\chi\chi)^{\kappa_2}[4\kappa_2(C_1^2 - C_2^2)]^{\kappa_2}\} = 0.$$

(ii) $\kappa_1 = 2/(m-1) \qquad \kappa_2 = 1/m \qquad m = 2, 3$

$$\begin{aligned}\psi(x) &= (1+\theta^2\omega^2)^{-(m+1)/2}\chi[\mathrm{i}C_1 + \gamma_4 C_2 - \theta(C_1 + \mathrm{i}\gamma_4 C_2)]\\ &\quad\times\begin{cases}(\gamma a)(ay) - (\gamma b)(by) - (\gamma c)(cy) & m=2\\ \gamma y & m=3\end{cases}\\ u(x) &= E(1+\theta^2\omega^2)^{(1-m)/2}\\ \omega &= \begin{cases}[(ay)^2 - (by)^2 - (cy)^2]^{1/2} & m=2\\ (yy)^{1/2} & m=3\end{cases}\end{aligned} \tag{19}$$

where θ, C_i and E are constants satisfying conditions

$$\theta^2(m^2-1)=[\mu_1|E|^{2/(m-1)}+\mu_2(\bar{\chi}\chi)^{1/m}(C_1^2-C_2^2)^{1/m}]^2$$

$$\theta(m+1)=[\lambda_1|E|^{2/(m-1)}+\lambda_2(\bar{\chi}\chi)^{1/m}(C_1^2-C_2^2)^{1/m}].$$

In (18), (19) we have used notations of (14).

References

[1] Heisenberg W 1954 *Z. Naturf.* a **9** 292
[2] Kortel F 1956 *Nuovo Cimento* **4** 210
[3] Akdeniz K G and Smailagic A 1979 *Nuovo Cimento* A **51** 345
[4] Merwe P T 1981 *Phys. Lett.* **106B** 485
[5] Kurdgelaidze D F 1959 *Zh. Eksp. Teor. Fiz.* **36** 842
[6] Fushchich W I and Shtelen W M 1983 *J. Phys. A: Math. Gen.* **16** 271
[7] Fushchich W I and Zhdanov R Z 1985 *Group Theoretical Studies of the Mathematical Physics Problems* (Kiev: Math. Inst.) p 20
[8] Fushchich W I and Zhdanov R Z 1987 *J. Phys. A: Math. Gen.* **20** 4173
[9] Gürsey F 1956 *Nuovo Cimento* **3** 988